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The ladder operator normal ordering problem for quantum confined systems and the generalization of the Stirling and Bell numbers

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Abstract
We solve the normal ordering problem for the ladder operators $\hat{B}_+$ and $\hat{B}_-$ for strings in the form $(\hat{B}_+^r\hat{B}_-^s)^n$ for supersymmetric and shape-invariant potential systems, with r, s and n positive integers. We provide exact and explicit expressions for their normal form $\mathcal{N}\{(\hat{B}_+^r\hat{B}_-^s)^n\}$, where in $\mathcal{N}\{\cdots\}$ all $\hat{B}_-$ are at the right side, and find that the solution involves an expansion coefficients sequence which, for $r, s > 1$, corresponds to the generalization, for any shape-invariant potential system, of the classical Stirling and Bell numbers. We show that these numbers are obtained for families of polynomials generated with parameters related to the forms of the supersymmetric partner potentials. We apply the general formalism to Pöschl–Teller, Morse, Scarf and self-similar potential systems.

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1. Introduction

The operator ordering problems frequently occur both in pure mathematics and theoretical physics studies. In many cases the mathematics is fairly complicated and it is often necessary to raise a function of non-commuting operators to certain powers or to rearrange operators in a certain order to facilitate calculations of trace and matrix elements. In physics, for the applications in quantum optics [1], studies in this field are restricted to the bosonic operators $\hat{a}$ and $\hat{a}^\dagger$ of the harmonic oscillator potential system, satisfying the commutation relation $[\hat{a}, \hat{a}^\dagger] = c\hat{I}$, where $\hat{I}$ is the identity operator and c a so-called c number. However, the harmonic oscillator is only the simplest example of a class of important potentials, called supersymmetric shape-invariant potentials [2], which are used with success in the models for quantum confined systems. Among these potentials we can cite also the Morse, Scarf, Pöschl–Teller, Hülthen and Coulomb potentials. The supersymmetric partner Hamiltonians $\hat{H}_{\pm} = \hat{p}_x^2 + V^{(\pm)}(x; a_1)$ for these systems can be written as $\hat{H}_- = \hat{A}^\dagger(a_1)\hat{A}(a_1)$ and $\hat{H}_+ = \hat{A}(a_1)\hat{A}^\dagger(a_1)$ where a_1 is a set of parameters, $\hat{A}(a_1) \equiv W(x; a_1) + \mathrm{i}\hat{p}_x$ and the superpotential $W(x; a_1)$ is a real function related to the partner potentials via $V^{(\pm)}(x; a_1) = W^2(x; a_1) \pm W'(x; a_1)$ (here dash denotes the derivative with respect to x and, for simplicity, in this study we take $\hbar = 2m = 1$). On the other hand, the shape invariance integrability condition [3], which characterize this set of potentials, is given by $\hat{A}(a_1)\hat{A}^\dagger(a_1) = \hat{A}^\dagger(a_2)\hat{A}(a_2) + R(a_1)$ where a_2 is a function of a_1 (say $a_2 = f(a_1)$) and the remainder $R(a_1)$ is independent of the dynamical variables.

Recently, the use of the operator techniques based on algebraic models [4–7] brought renewed interest to the study of shape-invariant systems. In [4] one of us introduced the parameter translation operator $T \equiv T(a_1)$ which acts in the a_n-potential parameters space $\mathcal{E}_a \equiv \{|a_n\rangle; n = 1, 2, 3, \ldots\}$ through the similarity transformation $\hat{T} O(a_1)\hat{T}^\dagger = O(a_2)$. With the ladder operators [4]

$$\hat{B}_+ = \hat{A}^\dagger(a_1)\hat{T} \qquad \text{and} \qquad \hat{B}_- = \hat{B}_+^\dagger = \hat{T}^\dagger\hat{A}(a_1), \tag{1}$$

the partner Hamiltonians $\hat{H}_{\pm}$ can be written as $\hat{H}_- = \hat{\mathcal{H}}_-$ and $\hat{H}_+ = \hat{T}\hat{\mathcal{H}}_+\hat{T}^\dagger$, where the *new* partner Hamiltonians are defined as $\hat{\mathcal{H}}_{\pm} = \hat{B}_{\mp}\hat{B}_{\pm}$, while the shape invariance condition is reduced to the ladder operator commutation relation

$$[\hat{B}_-, \hat{B}_+] = \hat{\mathcal{H}}_+ - \hat{\mathcal{H}}_- = \hat{T}^\dagger R(a_1)\hat{T} \equiv R(a_0), \tag{2}$$

suggesting that, indeed, the operators $\hat{B}_+$ and $\hat{B}_-$ are the appropriate creation and annihilation operators for the spectra of the shape-invariant potential systems provided that their non-commutativity with $R(a_n)$ is taken into account. From the remainder ladder relations

$$\hat{B}_{\pm}R(a_n) = R(a_{n\pm1})\hat{B}_{\pm} \tag{3}$$

which readily follows from the definition of $\hat{B}_{\pm}$, and the ground state annihilation condition $\hat{A}|0\rangle = 0 = \hat{B}_-|0\rangle$, it is possible to obtain the eigenvalue equations:

$$\hat{\mathcal{H}}_-|n\rangle = e_n|n\rangle \qquad \text{and} \qquad \hat{\mathcal{H}}_+|n\rangle = \{e_n + R(a_0)\}|n\rangle \tag{4}$$

where the normalized nth excited eigenstate $|n\rangle = \hat{\mathcal{K}}_+^n|0\rangle$ can be obtained from the ground state by the successive action of the raising operator $\hat{\mathcal{K}}_+ \equiv \frac{1}{\sqrt{\hat{\mathcal{H}}_-}}\hat{B}_+$ and the related eigenvalues are given by

$$e_0 = 0 \qquad \text{and} \qquad e_n = \sum_{k=1}^{n} R(a_k) \qquad \text{for} \quad n \geqslant 1. \tag{5}$$

With the results above, it is possible to show that

$$\hat{B}_+|n\rangle = \sqrt{e_{n+1}}|n+1\rangle \qquad \text{and} \qquad \hat{B}_-|n\rangle = \sqrt{e_{n-1} + R(a_0)}|n-1\rangle, \tag{6}$$

making clear the ladder nature of the operators $\hat{B}_{\pm}$ when applied on the eigenstates $\{|n\rangle; n = 0, 1, 2, \ldots\}$ of $\hat{\mathcal{H}}_{\pm}$. On the other hand, note that from the remainder ladder relations (3) one gets the commutation relations

$$\underbrace{[\hat{B}_+, [\hat{B}_+, [\hat{B}_+, \ldots, [\hat{B}_+, [\hat{B}_+, [\hat{B}_+, R_0]]] \ldots]]]}_{\text{sequence of } n \text{ commutation operations}} = \left\{\sum_{k=0}^{n} (-1)^k \binom{n}{k} R_{n-k}\right\}\hat{B}_+^n, \tag{7}$$

where we used the binomial coefficient definition: $\binom{n}{k} = n!/\{k!(n-k)!\}$ and assumed the simplifying notation $R_n \equiv R(a_n)$. There are an infinite number of these commutation relations which, with their adjoint commutation relations and the commutator (2), form an infinite-dimensional Lie algebra, realized here in a unitary representation.

Taking into account the algebraic formulation based on the ladder operators $\hat{B}_{\pm}$, we can consider these operators as the generalization of the boson operators for shape-invariant potential systems. In this paper, we begin the study of the normal ordering of these operators and obtain closed formulas generalizing the conventional combinatorial Bell and Stirling numbers related to the expansion coefficients of the normal ordering expressions.

The outline of the paper is the following. In section 2 we proceed with the normal ordering of the ladder operator string $(\hat{B}_{+}\hat{B}_{-})^{n}$ and obtain the shape-invariant system generalization of the conventional Stirling and Bell numbers. In sections 3 and 4 we do the same considering, respectively, the ladder operator strings $(\hat{B}_{+}^{r}\hat{B}_{-})^{n}$, $(\hat{B}_{+}\hat{B}_{-}^{s})^{n}$ and $(\hat{B}_{+}^{r}\hat{B}_{-}^{s})^{n}$ where r, s and n are positive integers. In section 5, we work out some examples and in section 6, we end the paper with our conclusions and brief remarks.

2. The ordering of the ladder operator string $(\hat{B}_{+}\hat{B}_{-})^{n}$

2.1. The ladder operator normal ordering

Any function $f(\hat{B}_{-}, \hat{B}_{+})$ of the ladder operators $\hat{B}_{-}$ and $\hat{B}_{+}$ can be defined by its power series expansion

$$f(\hat{B}_{-}, \hat{B}_{+}) = \sum_{k,l,m,n,\ldots} C_{klmn\ldots}\hat{B}_{-}^{k}\hat{B}_{+}^{l}\hat{B}_{+}^{m}\hat{B}_{-}^{n}\ldots. \tag{8}$$

Following the idea of the normal ordering procedure of the boson operators in the harmonic oscillator potential case [1, 8–10], by normal ordering process of $f(\hat{B}_{-}, \hat{B}_{+})$ we mean $\mathcal{N}[f(\hat{B}_{-}, \hat{B}_{+})]$ which is obtained by moving all the lowering operators $\hat{B}_{-}$ to the right in each term of expansion (8) using the commutation relation (2). In this process, the equality $f(\hat{B}_{-}, \hat{B}_{+}) = \mathcal{N}[f(\hat{B}_{-}, \hat{B}_{+})]$ must be satisfied. In other words, we say that the function $f(\hat{B}_{-}, \hat{B}_{+})$ is in normal order form if we can write

$$\mathcal{N}[f(\hat{B}_{-}, \hat{B}_{+})] = f(\hat{B}_{-}, \hat{B}_{+}) = \sum_{m,n} C_{mn}^{(\mathcal{N})}\hat{B}_{+}^{n}\hat{B}_{-}^{m} \tag{9}$$

after to use the shape invariance commutation relation (2).

Because of the infinite dimensional nature of the Lie algebra of the shape-invariant systems, the process of normal ordering of strings composed by the ladder operators $\hat{B}_{+}$ and $\hat{B}_{-}$ is harder than the harmonic oscillator potential case.

The main purpose of the following sections is to proceed with the normal ordering for the ladder operator string with the form $(\hat{B}_{+}^{r}\hat{B}_{-}^{s})^{n}$ for $r \geqslant s$ and $n \geqslant 1$. However, before that, we study some particular cases which are also interesting.

2.2. The shape-invariant generalization of the Stirling and Bell numbers

For this moment, we restrict ourselves to the case $r = s = 1$ in such way that we are searching for the normally ordered form of the nth power of the partner Hamiltonian $\hat{\mathcal{H}}_{-} = \hat{B}_{+}\hat{B}_{-}$. Note that this operator is the shape-invariant analog of the number operator $\hat{N} = \hat{a}^{\dagger}\hat{a}$ for the harmonic oscillator system. To make the procedure easier we first obtain an auxiliary relation. Taking into account the commutation relation (2), the remainder ladder relations (3) and the eigenvalue expression (5), one can show that

$$\begin{aligned} (\hat{B}_{+}\hat{B}_{-})\hat{B}_{+}\hat{B}_{-} &= \hat{B}_{+}^{2}\hat{B}_{-}^{2} + e_{1}\hat{B}_{+}\hat{B}_{-} \\ (\hat{B}_{+}\hat{B}_{-})\hat{B}_{+}^{2}\hat{B}_{-}^{2} &= \hat{B}_{+}^{3}\hat{B}_{-}^{3} + e_{2}\hat{B}_{+}^{2}\hat{B}_{-}^{2} \end{aligned} \tag{10}$$

and, in general, we can prove by induction that

$$(\hat{B}_+\hat{B}_-)\hat{B}_+^k\hat{B}_-^k = \hat{B}_+^{k+1}\hat{B}_-^{k+1} + e_k\hat{B}_+^k\hat{B}_-^k. \tag{11}$$

This relation enables us to compute the string $(\hat{B}_+\hat{B}_-)^n = (\hat{B}_+\hat{B}_-)(\hat{B}_+\hat{B}_-)^{n-1}$ since starting with $k = 1$, equation (11) yields

$$(\hat{B}_+\hat{B}_-)^2 = \hat{B}_+^2\hat{B}_-^2 + e_1\hat{B}_+\hat{B}_- \tag{12}$$

$$\begin{aligned}
(\hat{B}_+\hat{B}_-)^3 &= \hat{B}_+^3\hat{B}_-^3 + (e_2+e_1)\hat{B}_+^2\hat{B}_-^2 + e_1^2\hat{B}_+\hat{B}_- \\
(\hat{B}_+\hat{B}_-)^4 &= \hat{B}_+^4\hat{B}_-^4 + (e_3+e_2+e_1)\hat{B}_+^3\hat{B}_-^3 + \left[e_2(e_2+e_1)+e_1^2\right]\hat{B}_+^2\hat{B}_-^2 + e_1^3\hat{B}_+\hat{B}_- \\
(\hat{B}_+\hat{B}_-)^5 &= \hat{B}_+^5\hat{B}_-^5 + (e_4+e_3+e_2+e_1)\hat{B}_+^4\hat{B}_-^4 + \left[e_3(e_3+e_2+e_1)+e_2(e_2+e_1)+e_1^2\right]\hat{B}_+^3\hat{B}_-^3 \\
&\quad + \left[e_2^2(e_2+e_1)+e_1^2e_2+e_1^3\right]\hat{B}_+^2\hat{B}_-^2 + e_1^4\hat{B}_+\hat{B}_-.
\end{aligned} \tag{13}$$

Therefore, if we continue this procedure to obtain higher n power of $(\hat{B}_+\hat{B}_-)^n$, we can show, by induction, that it is possible to write

$$(\hat{B}_+\hat{B}_-)^n = \sum_{k=1}^{n} \mathcal{S}(k, n-k)\hat{B}_+^k\hat{B}_-^k, \tag{14}$$

where the expansion coefficients $\mathcal{S}(k, n-k)$, defined as

$$\mathcal{S}(k,0) = 1, \qquad \forall k \in \mathbb{Z} \qquad \text{and} \qquad \mathcal{S}(k,j) = \sum_{s=1}^{k} e_s\mathcal{S}(s, j-1) \qquad \text{for} \quad j,k \neq 0, \tag{15}$$

can be identified as the generalization, for any shape-invariant system, of the *classical Stirling numbers* $S(n, k)$ [11]. The classical Stirling numbers $S(n, k)$, related to the harmonic oscillator potential case, have an original combinatorial interpretation in terms of partitions of the set [12, 13]. They count the number of ways of putting n different objects into k identical containers, none left empty.

At this point it is also possible to define the *generalized Bell numbers* $\mathcal{B}(n)$ for shape-invariant systems, as well as the generalized *Bell function* $\mathcal{B}(n, x)$, through

$$\mathcal{B}(n) = \sum_{k=1}^{n} \mathcal{S}(k, n-k) \qquad \text{and} \qquad \mathcal{B}(n,x) = \sum_{k=1}^{n} \mathcal{S}(k, n-k)x^k. \tag{16}$$

The function $\mathcal{B}(n, x)$ is the shape-invariant system generalization of the conventional Bell polynomials $B(n, x)$ [14]. Also the *classical Bell numbers* $B(n)$ [11] have a combinatorial interpretation since they count the numbers of ways of putting n different objects into n identical containers, some may be left empty [12, 13]. Because of their increasing relevance as basic tools, the classical Stirling and Bell numbers are now textbook material in mathematical physics [15].

From definition (15), we can state the *recurrence relation* for the generalized Stirling numbers, which can also be obtained through

$$\mathcal{S}(k, j+1) = \sum_{s=1}^{k} e_s\mathcal{S}(s, j) = e_k\mathcal{S}(k, j) + \sum_{s=1}^{k-1} e_s\mathcal{S}(s, j),$$

which yields

$$\mathcal{S}(k, j+1) = \mathcal{S}(k-1, j+1) + e_k\mathcal{S}(k, j). \tag{17}$$

This recurrence relation, which corresponds to the shape-invariant system generalization of the standard recurrence relation for the classical Stirling numbers $S(n, k)$ [12, 13], can also be obtained in a longer way using relation (11) and expansion (14).

2.3. The shape-invariant generalization of the Dobiński-type relation

The eigenstates $\{|n\rangle; n = 0, 1, 2, \ldots\}$ of the partner Hamiltonians $\hat{\mathcal{H}}_\pm$ and the coherent states $|z; a_j\rangle$ are the most important sets of states within the Hilbert space of a shape-invariant system. The eigenstates satisfy the eigenvalue equation (4) while the coherent states, in their simplest unnormalized formulation [16, 17], satisfy the eigenvalue equations

$$\hat{B}_-|z; a_j\rangle = z|z; a_j\rangle \qquad \text{and} \qquad \langle z; a_j|\hat{B}_+ = \langle z; a_j|z^*. \tag{18}$$

These states satisfy the analog of the well-known Glauber expansion [16]

$$|z; a_j\rangle = \sum_{r=0}^{\infty} \frac{z^r}{h_r(a_j)}|r\rangle, \qquad \text{with} \qquad h_r(a_j) = \prod_{k=0}^{r-1} \sqrt{\mathcal{E}_r^{\{k\}}} \qquad \text{for} \quad r \geqslant 1 \tag{19}$$

and $h_0(a_j) = 1$, where

$$\mathcal{E}_r^{\{k\}} \equiv e_r - e_{k-1} = R_r + R_{r-1} + \cdots + R_{k+1} + R_k = \sum_{s=k}^{r} R_s \qquad \text{with} \quad k \leqslant r. \tag{20}$$

Note that from expression (19), we get

$$N(\rho, a_j) \equiv \langle z; a_j|z; a_j\rangle = \sum_{r=0}^{\infty} \left\{ \frac{\rho^r}{h_r^2(a_j)} \right\}, \tag{21}$$

where $\rho = |z|^2$, and the domain $(0 \leqslant \rho \leqslant R)$ of the allowed values for ρ is determined by the radius of convergence $\mathcal{R} = \limsup_{n\to+\infty} \sqrt[n]{h_n^2(a_j)}$ in the series above defining $N(\rho, a_j)$ [16].

Taking into account the eigenvalue equations (18), the evaluation of the expectation values of both sides of equation (14) with respect to the coherent state $|z; a_j\rangle$ yields

$$\begin{aligned} \langle z; a_j|(\hat{B}_+\hat{B}_-)^n|z; a_j\rangle &= \sum_{k=1}^{n} \mathcal{S}(k, n-k)\langle z; a_j|\hat{B}_+^k \hat{B}_-^k|z; a_j\rangle \\ &= N(\rho, a_j) \sum_{k=1}^{n} \mathcal{S}(k, n-k)\rho^k. \end{aligned} \tag{22}$$

Recalling $\hat{\mathcal{H}}_- = \hat{B}_+\hat{B}_-$ and considering the eigenvalue equations (4) in the Glauber expansion expression (19) we find that

$$\langle z; a_j|(\hat{B}_+\hat{B}_-)^n|z; a_j\rangle = \sum_{r=0}^{\infty} \left\{ \frac{\rho^r}{h_r^2(a_j)} \right\} e_r^n. \tag{23}$$

Therefore from (16), (19), (22) and (23), we obtain the shape-invariant system generalization of the celebrated *Dobiński formula* [11, 18, 19] for the Bell function

$$\mathcal{B}(n, \rho) = \frac{1}{N(\rho, a_j)} \sum_{r=0}^{\infty} \left\{ \frac{\rho^r}{h_r^2(a_j)} \right\} e_r^n \tag{24}$$

and, taking $\rho = 1$, the same for the Bell numbers $\mathcal{B}(n) \equiv \mathcal{B}(n, 1)$. These expressions assume definitive forms for each shape-invariant system, when the relation giving the remainder $R_n \equiv R(a_n)$ is specified. The most striking property of the Dobińsky formula is the representation of the number $\mathcal{B}(n)$ and the function $\mathcal{B}(n, \rho)$ as nontrivial infinite sums.

2.4. Procedure to obtain the generating function of the Bell function

A very elegant and efficient way of storing and tackling information about sequences is attained through their generating functions [11]. Inspired by the exponential function procedure used in the conventional Bell polynomials case [14], we present a possible way to obtain a generating function $\mathcal{G}(\lambda, x)$ of the Bell function $\mathcal{B}(n, x)$ by introducing the expansion

$$\mathcal{G}(\lambda, x) = \sum_{n=0}^{\infty} \mathcal{B}(n, x) \left\{ \frac{\lambda^n}{h_n^2(a_j)} \right\} \tag{25}$$

in such way that when we insert (24) into (25) and change the summation order we obtain

$$\begin{aligned}\mathcal{G}(\lambda, x) &= \frac{1}{N(x, a_j)} \sum_{n=0}^{\infty} \left[\sum_{r=0}^{\infty} \left\{ \frac{x^r}{h_r^2(a_j)} \right\} e_r^n \right] \left\{ \frac{\lambda^n}{h_n^2(a_j)} \right\} \\ &= \frac{1}{N(x, a_j)} \sum_{r=0}^{\infty} \left\{ \frac{x^r}{h_r^2(a_j)} \right\} \left[\sum_{n=0}^{\infty} \left\{ \frac{(\lambda e_r)^n}{h_n^2(a_j)} \right\} \right]\end{aligned}$$

that, from (21), reads

$$\mathcal{G}(\lambda, x) = \frac{1}{N(x, a_j)} \sum_{r=0}^{\infty} \left\{ \frac{x^r}{h_r^2(a_j)} \right\} N(\lambda e_r, a_j). \tag{26}$$

As in the case of the Dobiński relation, the definitive form of $\mathcal{G}(\lambda, x)$ for each shape-invariant system can be worked out after some calculations when the respective relation giving the remainder $R_n \equiv R(a_n)$ is considered. With the expression of $\mathcal{G}(\lambda, x)$, it is possible to obtain the shape-invariant system generalization of Sheffer identity [20] and the recurrence relation for the generalized Bell functions $\mathcal{B}(n, x)$.

We note that in the harmonic oscillator potential case we have $e_n = n$ and all the results obtained until now are reduced to the classical ones obtained in the literature for the study of the boson normal ordering problem [13, 14, 21–25]. We cite, as an important example, [26] where the solution of the normal ordering of the boson ladder operators is obtained through of a combinatorial method resulting in the generalization of the Stirling and Bell numbers, Bell polynomials and Dobiński relations.

3. The ordering of the ladder operator strings $(\hat{B}_+^r \hat{B}_-)^n$ and $(\hat{B}_+ \hat{B}_-^s)^n$

3.1. Auxiliary relations for $\hat{B}_- \hat{B}_+^r$ and $\hat{B}_-^s \hat{B}_+$

As a first step in the evaluation of the ladder operator strings $(\hat{B}_+^r \hat{B}_-)^n$ with $n, r > 1$, we obtain an auxiliary relation for $\hat{B}_- \hat{B}_+^r$ using (2) and (3):

$$\begin{aligned}\hat{B}_- \hat{B}_+^r &= (\hat{B}_- \hat{B}_+) \hat{B}_+^{r-1} \\ &= (\hat{B}_+ \hat{B}_- + R_0) \hat{B}_+^{r-1} \\ &= \hat{B}_+ (\hat{B}_- \hat{B}_+) \hat{B}_+^{r-2} + R_0 \hat{B}_+^{r-1} \\ &= \hat{B}_+ (\hat{B}_+ \hat{B}_- + R_0) \hat{B}_+^{r-2} + R_0 \hat{B}_+^{r-1} \\ &= \hat{B}_+^2 (\hat{B}_- \hat{B}_+) \hat{B}_+^{r-3} + (R_1 + R_0) \hat{B}_+^{r-1} \\ &= \hat{B}_+^2 (\hat{B}_+ \hat{B}_- + R_0) \hat{B}_+^{r-3} + (R_1 + R_0) \hat{B}_+^{r-1} \\ &\quad \vdots \\ &\quad \text{after } r\text{-steps}\end{aligned}$$

$$\vdots$$

$$= \hat{B}_+^{r-1}(\hat{B}_+\hat{B}_- + R_0)\hat{B}_+^0 + (R_{r-2} + R_{r-3} + \ldots + R_1 + R_0)\hat{B}_+^{r-1}$$
$$= \hat{B}_+^r\hat{B}_- + (R_{r-1} + R_{r-2} + R_{r-3} + \ldots + R_1 + R_0)\hat{B}_+^{r-1},$$

which gives

$$\hat{B}_-\hat{B}_+^r = \hat{B}_+^r\hat{B}_- + \mathcal{E}_{r-1}^{\{0\}}\hat{B}_+^{r-1} \tag{27}$$

where the factors $\mathcal{E}_n^{\{k\}}$ are defined in (20). Following the same sequence of steps above, it is possible to obtain

$$\hat{B}_-^s\hat{B}_+ = \hat{B}_+\hat{B}_-^s + \hat{B}_-^{s-1}\mathcal{E}_{s-1}^{\{0\}} \tag{28}$$

for $s > 1$. This relation can be used in the evaluation of the string $(\hat{B}_+\hat{B}_-^s)^n$.

3.2. *The shape-invariant generalized Stirling numbers $\mathcal{S}_{r,1}^{(n)}(k, n-k)$ and Bell numbers $\mathcal{B}_{r,1}(n)$*

Note that by the remainder ladder relations (3) and definition (20), we have the relations

$$\hat{B}_\pm^s\mathcal{E}_n^{\{k\}} = R_{n\pm s} + R_{n\pm s-1} + \cdots + R_{k\pm s+1} + R_{k\pm s} = \mathcal{E}_{n\pm s}^{\{k\pm s\}}\hat{B}_\pm^s. \tag{29}$$

Therefore, using relations (27) and (29) we can evaluate the strings $(\hat{B}_+^r\hat{B}_-)^n = \hat{B}_+^r(\hat{B}_-\hat{B}_+^r)^{n-1}\hat{B}_-$ and show that

$$(\hat{B}_+^r\hat{B}_-)^2 = \hat{B}_+^{2r}\hat{B}_-^2 + \mathcal{E}_{2r-1}^{\{r\}}\hat{B}_+^{2r-1}\hat{B}_- \tag{30}$$

$$\begin{aligned}(\hat{B}_+^r\hat{B}_-)^3 &= \hat{B}_+^{3r}\hat{B}_-^3 + \left(\mathcal{E}_{3r-1}^{\{r\}} + \mathcal{E}_{3r-2}^{\{2r-1\}}\right)\hat{B}_+^{3r-1}\hat{B}_-^2 + \mathcal{E}_{3r-2}^{\{r\}}\mathcal{E}_{3r-2}^{\{2r-1\}}\hat{B}_+^{3r-2}\hat{B}_-\\
(\hat{B}_+^r\hat{B}_-)^4 &= \hat{B}_+^{4r}\hat{B}_-^4 + \left(\mathcal{E}_{4r-1}^{\{r\}} + \mathcal{E}_{4r-2}^{\{2r-1\}} + \mathcal{E}_{4r-3}^{\{3r-2\}}\right)\hat{B}_+^{4r-1}\hat{B}_-^3 + \left[\mathcal{E}_{4r-2}^{\{r\}}\left(\mathcal{E}_{4r-2}^{\{2r-1\}} + \mathcal{E}_{4r-3}^{\{3r-2\}}\right)\right.\\
&\quad\left. + \mathcal{E}_{4r-3}^{\{2r-1\}}\mathcal{E}_{4r-3}^{\{3r-2\}}\right]\hat{B}_+^{4r-2}\hat{B}_-^2 + \mathcal{E}_{4r-3}^{\{r\}}\mathcal{E}_{4r-3}^{\{2r-1\}}\mathcal{E}_{4r-3}^{\{3r-2\}}\hat{B}_+^{4r-3}\hat{B}_-\end{aligned} \tag{31}$$

$$\begin{aligned}(\hat{B}_+^r\hat{B}_-)^5 &= \hat{B}_+^{5r}\hat{B}_-^5 + \left(\mathcal{E}_{5r-1}^{\{r\}} + \mathcal{E}_{5r-2}^{\{2r-1\}} + \mathcal{E}_{5r-3}^{\{3r-2\}} + \mathcal{E}_{5r-4}^{\{4r-3\}}\right)\hat{B}_+^{5r-1}\hat{B}_-^4\\
&\quad + \left[\mathcal{E}_{5r-2}^{\{r\}}\left(\mathcal{E}_{5r-2}^{\{2r-1\}} + \mathcal{E}_{5r-3}^{\{3r-2\}} + \mathcal{E}_{5r-4}^{\{4r-3\}}\right) + \mathcal{E}_{5r-3}^{\{2r-1\}}\left(\mathcal{E}_{5r-3}^{\{3r-2\}} + \mathcal{E}_{5r-4}^{\{4r-3\}}\right)\right.\\
&\quad\left. + \mathcal{E}_{5r-4}^{\{3r-2\}}\mathcal{E}_{5r-4}^{\{4r-3\}}\right]\hat{B}_+^{5r-2}\hat{B}_-^3 + \left\{\mathcal{E}_{5r-3}^{\{r\}}\left[\mathcal{E}_{5r-3}^{\{2r-1\}}\left(\mathcal{E}_{5r-3}^{\{3r-2\}} + \mathcal{E}_{5r-4}^{\{4r-3\}}\right) + \mathcal{E}_{5r-4}^{\{3r-2\}}\mathcal{E}_{5r-4}^{\{4r-3\}}\right]\right.\\
&\quad\left. + \mathcal{E}_{5r-4}^{\{2r-1\}}\mathcal{E}_{5r-4}^{\{3r-2\}}\mathcal{E}_{5r-4}^{\{4r-3\}}\right\}\hat{B}_+^{5r-3}\hat{B}_-^2 + \mathcal{E}_{5r-4}^{\{r\}}\mathcal{E}_{5r-4}^{\{2r-1\}}\mathcal{E}_{5r-4}^{\{3r-2\}}\mathcal{E}_{5r-4}^{\{4r-3\}}\hat{B}_+^{5r-4}\hat{B}_-.\end{aligned} \tag{32}$$

Note that if we define the factors

$$\mathcal{S}_{r,1}^{(n)}(n, j) = 1, \qquad \forall\, j \qquad \text{and}$$

$$\mathcal{S}_{r,1}^{(n)}(k, j) = \sum_{p=j}^{n-1}\mathcal{E}_{nr-p}^{\{(p+k-n+1)(r-1)+1\}}\mathcal{S}_{r,1}^{(n)}(k+1, p), \qquad \forall\, k < n, \tag{33}$$

and we continue the procedure used above to obtain higher power n of $(\hat{B}_+^r\hat{B}_-)^n$, we can show by induction that it is possible to write

$$(\hat{B}_+^r\hat{B}_-)^n = \sum_{k=1}^{n}\mathcal{S}_{r,1}^{(n)}(k, n-k)\hat{B}_+^{nr-n+k}\hat{B}_-^k \tag{34}$$

and, from this normally ordered expression, we can define the Stirling numbers $\mathcal{S}_{r,1}^{(n)}(k, n-k)$ that, in this case, are given by relation (33). Consequently, we can define also the Bell functions

$\mathcal{B}_{r,1}(n,x)$ and the Bell numbers $\mathcal{B}_{r,1}(n)$ as

$$\mathcal{B}_{r,1}(n,x) \equiv \sum_{k=1}^{n} \mathcal{S}_{r,1}^{(n)}(k,n-k)x^k \qquad \text{and} \qquad \mathcal{B}_{r,1}(n) \equiv \mathcal{B}_{r,1}(n,1) = \sum_{k=1}^{n} \mathcal{S}_{r,1}^{(n)}(k,n-k). \tag{35}$$

The numbers $\mathcal{S}_{r,1}^{(n)}(k,n-k)$ and $\mathcal{B}_{r,1}(n)$ are, respectively, the shape-invariant system generalization of the conventional Stirling and Bell numbers $S_{r,1}(n,k)$ and $B_{r,1}(n)$ found in the literature [13, 14].

From definitions (15) and (33), it is easy to verify that in the case of $r = 1$ we have the expected relation $\mathcal{S}_{1,1}^{(n)}(k,n-k) = \mathcal{S}(k,n-k)$ among the generalized Stirling numbers obtained in this and in the previous sections. On the other hand, for the case of $r = 2$, we obtain the Stirling numbers $\mathcal{S}_{2,1}^{(n)}(k,n-k)$, which is the shape-invariant system generalization of the classical so-called *unsigned Lah numbers* $L(n,k)$ [19].

To obtain the normally ordered form of the string $(\hat{B}_+\hat{B}_-^s)^n$ with $s > 1$, we can use the auxiliary relation (28) and the same induction approach of the previous case to show, after the same laborious calculation, that

$$(\hat{B}_+\hat{B}_-^s)^n = \sum_{k=1}^{n} \hat{B}_+^k \hat{B}_-^{ns-n+k} \mathcal{S}_{1,s}^{(n)}(k,n-k), \qquad \text{where} \tag{36}$$

$$\mathcal{S}_{1,s}^{(n)}(n,j) = 1, \qquad \forall\, j \qquad \text{and}$$

$$\mathcal{S}_{1,s}^{(n)}(k,j) = \sum_{p=j}^{n-1} \mathcal{E}_{ns-p}^{\{(p+k-n+1)(s-1)+1\}} \mathcal{S}_{1,s}^{(n)}(k+1,p), \qquad \forall\, k < n. \tag{37}$$

4. The ordering of the ladder operator string $(\hat{B}_+^r\hat{B}_-^s)^n$

4.1. Auxiliary relation for $\hat{B}_-^s\hat{B}_+^r$

Using result (27), we are able to obtain an expression for the term $\hat{B}_-^s\hat{B}_+^r = \hat{B}_-(\hat{B}_-^{s-1}\hat{B}_+^r)$ and use it to get a final expression for the normal ordering of the string $(\hat{B}_+^r\hat{B}_-^s)^n$, where $r > s$ and $s, n > 1$. We get

$$\hat{B}_-^2\hat{B}_+^r = \hat{B}_+^r\hat{B}_-^2 + \left(\mathcal{E}_{r-1}^{\{0\}} + \mathcal{E}_{r-2}^{\{-1\}}\right)\hat{B}_+^{r-1}\hat{B}_- + \mathcal{E}_{r-2}^{\{-1\}}\mathcal{E}_{r-2}^{\{0\}}\hat{B}_+^{r-2} \tag{38}$$

$$\begin{aligned}\hat{B}_-^3\hat{B}_+^r = {}& \hat{B}_+^r\hat{B}_-^3 + \left(\mathcal{E}_{r-1}^{\{0\}} + \mathcal{E}_{r-2}^{\{-1\}} + \mathcal{E}_{r-3}^{\{-2\}}\right)\hat{B}_+^{r-1}\hat{B}_-^2 + \left[\mathcal{E}_{r-2}^{\{0\}}\left(\mathcal{E}_{r-2}^{\{-1\}} + \mathcal{E}_{r-3}^{\{-2\}}\right)\right.\\ & \left. + \mathcal{E}_{r-3}^{\{-1\}}\mathcal{E}_{r-3}^{\{-2\}}\right]\hat{B}_+^{r-2}\hat{B}_- + \mathcal{E}_{r-3}^{\{-2\}}\mathcal{E}_{r-3}^{\{-1\}}\mathcal{E}_{r-3}^{\{0\}}\hat{B}_+^{r-3}\end{aligned} \tag{39}$$

$$\begin{aligned}\hat{B}_-^4\hat{B}_+^r = {}& \hat{B}_+^r\hat{B}_-^4 + \left(\mathcal{E}_{r-1}^{\{0\}} + \mathcal{E}_{r-2}^{\{-1\}} + \mathcal{E}_{r-3}^{\{-2\}} + \mathcal{E}_{r-4}^{\{-3\}}\right)\hat{B}_+^{r-1}\hat{B}_-^3\\ & + \left[\mathcal{E}_{r-2}^{\{0\}}\left(\mathcal{E}_{r-2}^{\{-1\}} + \mathcal{E}_{r-3}^{\{-2\}} + \mathcal{E}_{r-4}^{\{-3\}}\right) + \mathcal{E}_{r-3}^{\{-1\}}\left(\mathcal{E}_{r-3}^{\{-2\}} + \mathcal{E}_{r-4}^{\{-3\}}\right) + \mathcal{E}_{r-4}^{\{-2\}}\mathcal{E}_{r-4}^{\{-3\}}\right]\hat{B}_+^{r-2}\hat{B}_-^2\\ & + \left\{\mathcal{E}_{r-3}^{\{0\}}\left[\mathcal{E}_{r-3}^{\{-1\}}\left(\mathcal{E}_{r-3}^{\{-2\}} + \mathcal{E}_{r-4}^{\{-3\}}\right) + \mathcal{E}_{r-4}^{\{-2\}}\mathcal{E}_{r-4}^{\{-3\}}\right] + \mathcal{E}_{r-4}^{\{-1\}}\mathcal{E}_{r-4}^{\{-2\}}\mathcal{E}_{r-4}^{\{-3\}}\right\}\hat{B}_+^{r-3}\hat{B}_-\\ & + \mathcal{E}_{r-4}^{\{0\}}\mathcal{E}_{r-4}^{\{-1\}}\mathcal{E}_{r-4}^{\{-2\}}\mathcal{E}_{r-4}^{\{-3\}}\hat{B}_+^{r-4}\end{aligned} \tag{40}$$

$$\begin{aligned}\hat{B}_-^5\hat{B}_+^r = {}& \hat{B}_+^r\hat{B}_-^5 + \left(\mathcal{E}_{r-1}^{\{0\}} + \mathcal{E}_{r-2}^{\{-1\}} + \mathcal{E}_{r-3}^{\{-2\}} + \mathcal{E}_{r-4}^{\{-3\}} + \mathcal{E}_{r-5}^{\{-4\}}\right)\hat{B}_+^{r-1}\hat{B}_-^4\\ & + \left[\mathcal{E}_{r-2}^{\{0\}}\left(\mathcal{E}_{r-2}^{\{-1\}} + \mathcal{E}_{r-3}^{\{-2\}} + \mathcal{E}_{r-4}^{\{-3\}} + \mathcal{E}_{r-5}^{\{-4\}}\right) + \mathcal{E}_{r-3}^{\{-1\}}\left(\mathcal{E}_{r-3}^{\{-2\}} + \mathcal{E}_{r-4}^{\{-3\}} + \mathcal{E}_{r-5}^{\{-4\}}\right)\right.\\ & \left. + \mathcal{E}_{r-4}^{\{-2\}}\left(\mathcal{E}_{r-4}^{\{-3\}} + \mathcal{E}_{r-5}^{\{-4\}}\right) + \mathcal{E}_{r-5,-3}\mathcal{E}_{r-5,-4}\right]\hat{B}_+^{r-2}\hat{B}_-^3\end{aligned}$$

$$+\left\{\mathcal{E}_{r-3}^{\{0\}}\left[\mathcal{E}_{r-3}^{\{-1\}}\left(\mathcal{E}_{r-3}^{\{-2\}}+\mathcal{E}_{r-4}^{\{-3\}}+\mathcal{E}_{r-5}^{\{-4\}}\right)+\mathcal{E}_{r-4}^{\{-2\}}\left(\mathcal{E}_{r-4}^{\{-3\}}+\mathcal{E}_{r-5}^{\{-4\}}\right)+\mathcal{E}_{r-5}^{\{-3\}}\mathcal{E}_{r-5}^{\{-4\}}\right]\right.$$
$$+\mathcal{E}_{r-4,-1}\left[\mathcal{E}_{r-4}^{\{-2\}}\left(\mathcal{E}_{r-4}^{\{-3\}}+\mathcal{E}_{r-5}^{\{-4\}}\right)+\mathcal{E}_{r-5}^{\{-3\}}\mathcal{E}_{r-5}^{\{-4\}}\right]+\left.\mathcal{E}_{r-5}^{\{-2\}}\mathcal{E}_{r-5}^{\{-3\}}\mathcal{E}_{r-5}^{\{-4\}}\right\}\hat{B}_+^{r-3}\hat{B}_-^2$$
$$+\left\{\mathcal{E}_{r-4}^{\{0\}}\mathcal{E}_{r-4}^{\{-1\}}\left[\mathcal{E}_{r-4}^{\{-2\}}\left(\mathcal{E}_{r-4}^{\{-3\}}+\mathcal{E}_{r-5}^{\{-4\}}\right)+\mathcal{E}_{r-5}^{\{-3\}}\mathcal{E}_{r-5}^{\{-4\}}\right]\right.$$
$$+\left.\mathcal{E}_{r-5}^{\{-1\}}\mathcal{E}_{r-5}^{\{-2\}}\mathcal{E}_{r-5}^{\{-3\}}\mathcal{E}_{r-5}^{\{-4\}}\right\}\hat{B}_+^{r-4}\hat{B}_-+\mathcal{E}_{r-5}^{\{0\}}\mathcal{E}_{r-5}^{\{-1\}}\mathcal{E}_{r-5}^{\{-2\}}\mathcal{E}_{r-5}^{\{-3\}}\mathcal{E}_{r-5}^{\{-4\}}\hat{B}_+^{r-5}. \quad (41)$$

At this point, if we define the factors

$$\mathcal{C}_r^{(s)}(0,j)=1, \quad \forall\, j \qquad \text{and} \qquad \mathcal{C}_r^{(s)}(k,j)=\sum_{p=j}^{s}\mathcal{E}_{r-p}^{\{k-p\}}\mathcal{C}_r^{(s)}(k-1,p), \quad \forall\, k>0, \quad (42)$$

we can write these results in general as

$$\hat{B}_-^s\hat{B}_+^r=\sum_{k=0}^{s}\mathcal{C}_r^{\{s\}}(k,k)\hat{B}_+^{r-k}\hat{B}_-^{s-k}, \qquad \text{with} \quad r>s. \quad (43)$$

When $r<s$, we can use result (28), relation (29), and, following the same sequence of steps as above, we can show that the auxiliary relation, in this case, will assume the form

$$\hat{B}_-^s\hat{B}_+^r=\sum_{k=0}^{r}\hat{B}_+^{r-k}\hat{B}_-^{s-k}\tilde{\mathcal{C}}_s^{\{r\}}(k,k), \quad (44)$$

where

$$\tilde{\mathcal{C}}_s^{(r)}(0,j)=1, \quad \forall\, j \qquad \text{and} \qquad \tilde{\mathcal{C}}_s^{(r)}(k,j)=\sum_{p=j}^{r}\mathcal{E}_{s-p}^{\{k-p\}}\tilde{\mathcal{C}}_s^{(r)}(k-1,p), \qquad \forall\, k>0. \quad (45)$$

The auxiliary relation in (44) can be used in the evaluation of the string $(\hat{B}_+^r\hat{B}_-^s)^n$ when $r<s$ and $r,n>1$.

4.2. The shape-invariant generalized Stirling numbers $\mathcal{S}_{r,s}^{(n)}(k,n-k)$ and Bell numbers $\mathcal{B}_{r,s}(n)$

At this point we can use relations (29) and (42) to show that

$$\hat{B}_\pm^n\mathcal{C}_r^{\{s\}}(k,j)=\mathcal{C}_r^{\{s\mp n\}}(k,j\mp n)\hat{B}_\pm^n \quad (46)$$

and taking into account this relation and (43) we can evaluate the string

$$\begin{aligned}(\hat{B}_+^r\hat{B}_-^s)^2&=\hat{B}_+^r(\hat{B}_-^s\hat{B}_+^r)\hat{B}_-^s\\&=\hat{B}_+^r\left(\sum_{k=0}^{s}\mathcal{C}_r^{\{s\}}(k,k)\hat{B}_+^{r-k}\hat{B}_-^{s-k}\right)\hat{B}_-^s\\&=\sum_{k=0}^{s}\mathcal{C}_r^{\{s-r\}}(k,k-r)\hat{B}_+^{2r-k}\hat{B}_-^{2s-k},\end{aligned} \quad (47)$$

which can be written as

$$(\hat{B}_+^r\hat{B}_-^s)^2=\sum_{k=0}^{s}\mathcal{S}_{r,s}^{(2)}(k,2-k)\hat{B}_+^{2r-k}\hat{B}_-^{2s-k} \qquad \text{where} \quad \mathcal{S}_{r,s}^{(2)}(k,2-k)=\mathcal{C}_r^{\{s-r\}}(k,k-r). \quad (48)$$

With the result obtained above and relations (43) and (46), we can evaluate

$$(\hat{B}_+^r\hat{B}_-^s)^3 = \hat{B}_+^r\hat{B}_-^s(\hat{B}_+^r\hat{B}_-^s)^2 = \sum_{k_1=0}^{s}\sum_{k_2=0}^{s}\mathcal{C}_r^{\{2(s-r)\}}(k_1, k_1-2r+s)\mathcal{C}_{2r-k_1}^{\{s-r\}}(k_2, k_2-r)\hat{B}_+^{3r-k_1-k_2}\hat{B}_-^{3s-k_1-k_2}. \quad (49)$$

Collecting the terms with the same $(k_1 + k_2)$-values in the powers of $\hat{B}_\pm$ and putting them together is possible to rewrite (49) as

$$(\hat{B}_+^r\hat{B}_-^s)^3 = \sum_{k=0}^{2s}\mathcal{S}_{r,s}^{(3)}(k, 3-k)\hat{B}_+^{3r-k}\hat{B}_-^{3s-k}, \quad (50)$$

where

$$\mathcal{S}_{r,s}^{(3)}(k, 3-k) = \sum_{k_1=0}^{k}\mathcal{C}_r^{\{2(s-r)\}}(k_1, k_1-2r+s)\mathcal{C}_{2r-k_1}^{\{s-r\}}(k-k_1, k-k_1-r). \quad (51)$$

As a next step, we can use equations (49) and (46) and the same procedure of the previous $n = 3$ power to evaluate the $n = 4$ power case and show that

$$\begin{aligned}(\hat{B}_+^r\hat{B}_-^s)^4 &= \hat{B}_+^r\hat{B}_-^s(\hat{B}_+^r\hat{B}_-^s)^3 \\ &= \sum_{k_1=0}^{s}\sum_{k_2=0}^{s}\sum_{k_3=0}^{s}\mathcal{C}_r^{\{3(s-r)\}}(k_1, k_1-3r+2s)\mathcal{C}_{2r-k_1}^{\{2(s-r)\}}(k_2, k_2-2r+s) \\ &\quad\times \mathcal{C}_{3r-k_1-k_2}^{\{s-r\}}(k_3, k_3-r)\hat{B}_+^{4r-k_1-k_2-k_3}\hat{B}_-^{4s-k_1-k_2-k_3}. \end{aligned} \quad (52)$$

After we collect the terms with the same $(k_1 + k_2 + k_3)$-values in the powers of $\hat{B}_\pm$ and to put them together, it is possible to rewrite (52) as

$$(\hat{B}_+^r\hat{B}_-^s)^4 = \sum_{k=0}^{3s}\mathcal{S}_{r,s}^{(4)}(k, 4-k)\hat{B}_+^{4r-k}\hat{B}_-^{4s-k}, \quad (53)$$

where

$$\begin{aligned}\mathcal{S}_{r,s}^{(4)}(k, 4-k) &= \sum_{k_1=0}^{k}\sum_{k_2=0}^{k-k_1}\mathcal{C}_r^{\{3(s-r)\}}(k_1, k_1-3r+2s)\mathcal{C}_{2r-k_1}^{\{2(s-r)\}}(k_2, k_2-2r+s) \\ &\quad\times \mathcal{C}_{3r-k_1-k_2}^{\{s-r\}}(k-k_1-k_2, k-k_1-k_2-r). \end{aligned} \quad (54)$$

As a last step, taking into account equations (52), (46) and following the same procedure as the previous power cases we obtain, after some calculations, the expression

$$(\hat{B}_+^r\hat{B}_-^s)^5 = \sum_{k=0}^{4s}\mathcal{S}_{r,s}^{(5)}(k, 5-k)\hat{B}_+^{5r-k}\hat{B}_-^{5s-k} \quad (55)$$

where

$$\begin{aligned}\mathcal{S}_{r,s}^{(5)}(k, 5-k) &= \sum_{k_1=0}^{k}\sum_{k_2=0}^{k-k_1}\sum_{k_3=0}^{k-k_1-k_2}\mathcal{C}_r^{\{4(s-r)\}}(k_1, k_1-4r+3s)\mathcal{C}_{2r-k_1}^{\{3(s-r)\}}(k_2, k_2-3r+2s) \\ &\quad\times \mathcal{C}_{3r-k_1-k_2}^{\{2(s-r)\}}(k_3, k_3-2r+s) \\ &\quad\times \mathcal{C}_{4r-k_1-k_2-k_3}^{\{s-r\}}(k-k_1-k_2-k_3, k-k_1-k_2-k_3-r). \end{aligned} \quad (56)$$

Continuing this procedure for higher power n of the string $(\hat{B}_+^r\hat{B}_-^s)^n$ we can show by induction that

$$(\hat{B}_+^r\hat{B}_-^s)^n = \sum_{k=0}^{ns-s} \mathcal{S}_{r,s}^{(n)}(k, n-k)\hat{B}_+^{nr-k}\hat{B}_-^{ns-k} \tag{57}$$

where

$$\mathcal{S}_{r,s}^{(n)}(k, n-k) = \underbrace{\sum_{k_1=0}^{k}\sum_{k_2=0}^{k-k_1}\sum_{k_3=0}^{k-k_1-k_2}\cdots\sum_{k_{n-2}=0}^{k-k_1-k_2-\ldots-k_{n-3}}}_{\text{sequence of } (n-2) \text{ sum operations}} \mathcal{C}_r^{\{(n-1)(s-r)\}}(k_1, k_1-(n-1)r$$
$$+(n-2)s)\mathcal{C}_{2r-k_1}^{\{(n-2)(s-r)\}}(k_2, k_2-(n-2)r+(n-3)s)$$
$$\times\,\mathcal{C}_{3r-k_1-k_2}^{\{(n-3)(s-r)\}}(k_3, k_3-(n-3)r+(n-4)s)$$
$$\times\,\mathcal{C}_{(n-1)r-k_1-k_2-\ldots-k_{n-2}}^{\{s-r\}}(k-k_1-k_2-\cdots-k_{n-2}, k-k_1-k_2-\cdots-k_{n-2}-r). \tag{58}$$

Considering the expression obtained in (57) for the string $(\hat{B}_+^r\hat{B}_-^s)^n$, we can identify the factors $\mathcal{S}_{r,s}^{(n)}(k, n-k)$, given in (58), as the generalization for any shape-invariant potential system of the classical Stirling numbers of second order $S_{r,s}(n, k)$ [14]. Consequently we can define the corresponding generalized Bell functions $\mathcal{B}_{r,s}(n, x)$ and Bell numbers $\mathcal{B}_{r,s}(n)$ as

$$\mathcal{B}_{r,s}(n, x) \equiv \sum_{k=1}^{n} \mathcal{S}_{r,s}^{(n)}(k, n-k)x^k \qquad \text{and} \qquad \mathcal{B}_{r,s}(n) \equiv \mathcal{B}_{r,s}(n, 1) = \sum_{k=1}^{n} \mathcal{S}_{r,s}^{(n)}(k, n-k). \tag{59}$$

When $r < s$ we can use (29) and (45) to obtain the ladder relation

$$\hat{B}_\pm^n\tilde{\mathcal{C}}_s^{\{r\}}(k, j) = \tilde{\mathcal{C}}_s^{\{r\mp n\}}(k, j\mp n)\hat{B}_\pm^n. \tag{60}$$

Taking into account this relation, (44), (45) and the same induction approach used before, it is possible to show that

$$(\hat{B}_+^r\hat{B}_-^s)^n = \sum_{k=0}^{nr-r} \hat{B}_+^{nr-k}\hat{B}_-^{ns-k}\tilde{\mathcal{S}}_{r,s}^{(n)}(k, n-k), \qquad \text{where} \quad \tilde{\mathcal{S}}_{r,s}^{(n)}(k, n-k) = \mathcal{S}_{s,r}^{(n)}(k, n-k). \tag{61}$$

From expansion (61) we identify the factors $\tilde{\mathcal{S}}_{r,s}^{(n)}(k, n-k)$ as the shape-invariant-system generalized Stirling numbers of second order when $r < s$. Thus we can define the corresponding generalized Bell functions $\tilde{\mathcal{B}}_{r,s}(n, x)$ and Bell numbers $\tilde{\mathcal{B}}_{r,s}(n)$ as

$$\tilde{\mathcal{B}}_{r,s}(n, x) \equiv \sum_{k=1}^{n} \tilde{\mathcal{S}}_{r,s}^{(n)}(k, n-k)x^k \qquad \text{and} \qquad \tilde{\mathcal{B}}_{r,s}(n) \equiv \tilde{\mathcal{B}}_{r,s}(n, 1) = \sum_{k=1}^{n} \tilde{\mathcal{S}}_{r,s}^{(n)}(k, n-k). \tag{62}$$

Comparing expressions (59) and (62), the symmetry relation $\tilde{\mathcal{B}}_{r,s}(n) = \mathcal{B}_{s,r}(n)$ is evident.

5. Applications to some shape-invariant potential systems

In order to illustrate how our general results can be applied in specific cases, we consider in this section the normal ordering ladder operators problem for three examples of shape-invariant potential systems: the first two presenting the potential parameters a_n related by translation and the third one with these parameters related by scaling.

5.1. The Pöschl–Teller potential system

The Pöschl–Teller potential [27] was originally introduced in a molecular physics context and is closely related to several other potentials widely used in molecular and solid state physics, such as, for example, the Scarf potential, used in the modeling of 1D crystal, and the Rosen–Morse potential, used in molecular models. Besides that, the Pöschl–Teller potential, in its trigonometric form, presents the interesting property of representing the infinite square-well as a special limit. The supersymmetric partner potentials $V^{(\pm)}(x; a_1) = -(a_1+\delta)^2 + a_1(a_1 \pm \alpha)\sec^2 \alpha x + \delta(\delta \pm \alpha)\csc^2 \alpha x$ for a trigonometric Pöschl–Teller potential system are obtained with the superpotential $W(x; a_1) = a_1 \tan \alpha x - \delta \cot \alpha x$, where a_1, α and δ are real constants and $\left(0 \leqslant \alpha x \leqslant \frac{\pi}{2}\right)$. In this case the potential parameters are related by $a_n = a_{n-1} + \alpha$ and the remainders involved with the shape invariance condition (2) are given by the recurrence relation $R_n = R_1 + 2\kappa(n-1)$, where $R_1 = \kappa(\gamma+1)$ with $\kappa = 4\alpha^2$ and $\gamma = (a_1+\delta)/\alpha$. Using these relations in (5) and (20) and setting the energy scale so that $\kappa = 1$ (as the usual procedure used in the harmonic oscillator case is) we find the eigenvalue factors e_n and $\mathcal{E}_r^{\{k\}}$ with the forms

$$e_n = n(\gamma + n) \qquad \text{and} \qquad \mathcal{E}_r^{\{k\}} = (r-k+1)\gamma + r^2 - (k-1)^2. \tag{63}$$

From expressions for R_n, e_n and $\mathcal{E}_r^{\{k\}}$ and their relations with the shape-invariant system generalized Stirling and Bell numbers through of equations (15), (16), (33), (35), (42), (58) and (59), we find these numbers given by polynomials with the form

$$\mathcal{S}_{r,s}^{(n)}(k, n-k) = c_{r,s}^{(n,0)}(k) + c_{r,s}^{(n,1)}(k)\gamma + c_{r,s}^{(n,2)}(k)\gamma^2 + \cdots + c_{r,s}^{(n,n-k)}(k)\gamma^{n-k} \tag{64}$$

$$\mathcal{B}_{r,s}(n) = \sum_{k=0}^{n-1} \left[c_{r,s}^{(n,0)}(k) + c_{r,s}^{(n,1)}(k)\gamma + c_{r,s}^{(n,2)}(k)\gamma^2 + \cdots + c_{r,s}^{(n,n-k)}(k)\gamma^{n-k}\right] \tag{65}$$

in such way that their numerical values are dependent on the value of the factor γ. Some examples illustrating this behaviour are given by

$$\mathcal{S}_{1,1}^{(5)}(5,0) = 1, \qquad \mathcal{S}_{1,1}^{(5)}(4,1) = 30 + 10\gamma, \qquad \mathcal{S}_{1,1}^{(5)}(3,2) = 147 + 120\gamma + 25\gamma^2,$$
$$\mathcal{S}_{1,1}^{(5)}(2,3) = 85 + 141\gamma + 79\gamma^2 + 15\gamma^3, \qquad \mathcal{S}_{1,1}^{(5)}(1,4) = 1 + 4\gamma + 6\gamma^2 + 4\gamma^3 + \gamma^4.$$

Therefore, we conclude that the factor γ defines the families of Stirling and Bell numbers for the Pöschl–Teller potential. Since γ defines the form of the Pöschl–Teller potential, the form of the potential is responsible for the definition of the family of $\mathcal{S}_{r,s}^{(n)}(k, n-k)$ and $\mathcal{B}_{r,s}(n)$ numbers. We note that since γ can assume any continuous value within its physically allowed range, there is an infinite number of possible families of Stirling and Bell numbers for Pöschl–Teller potential systems. The common x dependence of the potentials with different forms is responsible for the common set of polynomial coefficients $\left\{c_{r,s}^{(n,j)}(k);\ j = 0, 1, 2, \ldots, n-k\right\}$ defining the elements $\mathcal{S}_{r,s}^{(n)}(k, n-k)$ and $\mathcal{B}_{r,s}(n)$ which belong to different families (γ values). Also this common x potential dependence is related to the common polynomial form of the numbers $\mathcal{S}_{r,s}^{(n)}(k, n-k)$ with the same $n-k$ value and different n and k values, as we can see comparing the Stirling numbers

$$\mathcal{S}_{1,1}^{(5)}(2,3) = 85 + 141\gamma + 79\gamma^2 + 15\gamma^3 \qquad \text{and} \qquad \mathcal{S}_{1,1}^{(4)}(1,3) = 1 + 3\gamma + 79\gamma^2 + \gamma^3. \tag{66}$$

Table 1. Triangle of the generalized polynomial Stirling numbers $\mathcal{S}_{1,1}^{(n)}(k, n-k)$ for a Pöschl–Teller potential system, given in the vector notation (67).

$r=1,\ s=1$	$\mathcal{S}_{1,1}^{(n)}(k,n-k), \quad 0 \le k \le n-1$					
$n=1$	$[1]$					
$n=2$	$[1]$	$\begin{bmatrix}1\\1\end{bmatrix}$				
$n=3$	$[1]$	$\begin{bmatrix}5\\3\end{bmatrix}$	$\begin{bmatrix}1\\2\\1\end{bmatrix}$			
$n=4$	$[1]$	$\begin{bmatrix}14\\6\end{bmatrix}$	$\begin{bmatrix}21\\24\\7\end{bmatrix}$	$\begin{bmatrix}1\\3\\3\\1\end{bmatrix}$		
$n=5$	$[1]$	$\begin{bmatrix}30\\10\end{bmatrix}$	$\begin{bmatrix}147\\120\\25\end{bmatrix}$	$\begin{bmatrix}85\\141\\79\\15\end{bmatrix}$	$\begin{bmatrix}1\\4\\6\\4\\1\end{bmatrix}$	
$n=6$	$[1]$	$\begin{bmatrix}55\\15\end{bmatrix}$	$\begin{bmatrix}627\\400\\65\end{bmatrix}$	$\begin{bmatrix}1408\\1662\\664\\90\end{bmatrix}$	$\begin{bmatrix}341\\738\\604\\222\\31\end{bmatrix}$	$\begin{bmatrix}1\\5\\10\\10\\5\\1\end{bmatrix}$

We conclude that it is sufficient to specify a given Stirling numbers family to present the numerical values of the polynomial coefficients associated with this family. Consequently we use the following simplifying vector notation after this point:

$$\mathcal{S}_{r,s}^{(n)}(k, n-k) \Longleftrightarrow \begin{bmatrix} c_{r,s}^{(n,0)} \\ c_{r,s}^{(n,1)} \\ c_{r,s}^{(n,2)} \\ \vdots \\ c_{r,s}^{(n,n-k)} \end{bmatrix}. \tag{67}$$

From the number $n-k+1$ of elements of the vector, we obtain the order $n-k$ of the polynomial for $\mathcal{S}_{r,s}^{(n)}(k, n-k)$.

In the tables 1, 2 and 3 we present, in the vector notation (67), the polynomial giving the Stirling $\mathcal{S}_{r,s}^{(n)}(k, n-k)$ numbers for some values of the powers n, r and s belonging to a generic γ-family of the Pöschl–Teller potential system.

To conclude this application, we note that the trigonometric Scarf potential [2], the superpotential of which is given by $W(x; a_1) = a_1 \tan \alpha x + \delta \sec \alpha x$, where a_1, α and δ are real constants and $\left(-\frac{\pi}{2} \leqslant \alpha x \leqslant \frac{\pi}{2}\right)$, has the eigenvalue factors e_n and $\mathcal{E}_r^{\{k\}}$ and the remainders R_n involved with the shape invariance condition (2) with the same forms presented for the Pöschl–Teller potential case. Therefore, the families of Stirling $\mathcal{S}_{r,s}^{(n)}(k, n-k)$ numbers for this potential have the same polynomial structure presented in the tables 1, 2 and 3.

Table 2. The same as table 1 for the generalized polynomial Stirling numbers $\mathcal{S}^{(n)}_{2,1}(k, n-k)$.

$r = 2,\ s = 1$	$\mathcal{S}^{(n)}_{2,1}(k, n-k), \quad 0 \le k \le n-1$				
$n = 1$	$[1]$				
$n = 2$	$[1]$	$\begin{bmatrix}8\\2\end{bmatrix}$			
$n = 3$	$[1]$	$\begin{bmatrix}36\\6\end{bmatrix}$	$\begin{bmatrix}180\\66\\6\end{bmatrix}$		
$n = 4$	$[1]$	$\begin{bmatrix}96\\12\end{bmatrix}$	$\begin{bmatrix}2016\\540\\36\end{bmatrix}$	$\begin{bmatrix}8064\\3504\\504\\24\end{bmatrix}$	
$n = 5$	$[1]$	$\begin{bmatrix}200\\20\end{bmatrix}$	$\begin{bmatrix}10800\\2280\\120\end{bmatrix}$	$\begin{bmatrix}172800\\58080\\6480\\240\end{bmatrix}$	$\begin{bmatrix}604800\\289680\\51720\\4080\\120\end{bmatrix}$

Table 3. The same as table 1 for the generalized polynomial Stirling numbers $\mathcal{S}^{(n)}_{3,2}(k, n-k)$.

$r = 3,\ s = 2$	$\mathcal{S}^{(n)}_{3,2}(k, n-k), \quad 0 \le k \le 2(n-1)$				
$n = 1$	$[1]$				
$n = 2$	$[1]$	$\begin{bmatrix}36\\6\end{bmatrix}$	$\begin{bmatrix}180\\66\\6\end{bmatrix}$		
$n = 3$	$[1]$	$\begin{bmatrix}156\\18\end{bmatrix}$	$\begin{bmatrix}6336\\1560\\96\end{bmatrix}$	$\begin{bmatrix}330576\\125040\\15840\\672\end{bmatrix}$	$\begin{bmatrix}18037440\\10317000\\2258856\\223656\\8424\end{bmatrix}$

5.2. The Morse potential system

The Morse potential [28] was originally introduced as a useful model for the diatomic molecules and has been widely used in the study of molecular vibrations, laser chemistry and, in particular, chemical bonds. Anharmonicities and dissociation effects, which may arise in a more realistic physical situation, are better represented using a Morse potential. The supersymmetric partner potentials $V^{(\pm)}(x; a_1) = a_1^2 + \mathrm{e}^{-2\alpha x} - (2a_1 \mp \alpha)\,\mathrm{e}^{-\alpha x}$ for a Morse potential system are obtained with the superpotential $W(x; a_1) = a_1 - \mathrm{e}^{-\alpha x}$, where a_1 and α are real constants. In this case the potential parameters are related by $a_n = a_{n-1} - \alpha$ and the remainders involved with the shape invariance condition (2) are given by the recurrence relation $R_n = R_1 - 2\kappa(n-1)$, where $R_1 = \kappa(\gamma - 1)$ with $\kappa = \alpha^2$ and $\gamma = 2a_1/\alpha$. Using these relations in (5) and (20) and setting $\kappa = 1$, we find the eigenvalue factors e_n and $\mathcal{E}_r^{\{k\}}$ as

$$e_n = n(\gamma - n) \qquad \text{and} \qquad \mathcal{E}_r^{\{k\}} = (r-k+1)\gamma - r^2 + (k-1)^2. \tag{68}$$

Because of the resemblance between these factors and those of the Pöschl–Teller potential system we find that the families of Stirling $\mathcal{S}_{r,s}^{(n)}(k, n-k)$ numbers for the Morse potential system have the same polynomial expressions (64) and (65), but with some coefficients $c_{r,s}^{(n,j)}(k)$ taking negative values: when k is even (odd) the coefficients of the terms γ^j with j even (odd) take negative values. Taking into account this sign modification in these terms, the polynomial giving the Stirling $\mathcal{S}_{r,s}^{(n)}(k, n-k)$, in the vector notation (67), for some values of the powers n, r and s belonging to a generic γ-family of Morse potential system can be obtained in tables 1, 2 and 3.

Two other examples of shape-invariant potential systems that share the families of the Stirling $\mathcal{S}_{r,s}^{(n)}(k, n-k)$ and the Bell $\mathcal{B}_{r,s}(n)$ numbers with the same polynomial structure of the Morse potential case are: (i) the hyperbolic Scarf potential [2], the superpotential of which is given by $W(x; a_1) = a_1 \tanh\alpha x + \delta\,\mathrm{sech}\,\alpha x$, and (ii) the generalized Pöschl–Teller potential [2], the superpotential of which is given by $W(x; a_1) = a_1 \coth\alpha x - \delta\,\mathrm{cosech}\,\alpha x$, where a_1, α and δ are real constants in both of the cases. Consequently, these potentials exhibit eigenvalue factors e_n and $\mathcal{E}_r^{\{k\}}$ and the remainders R_n involved in the shape invariance condition (2) in the same forms as that of the Morse potential case.

5.3. The self-similar potential system

Another class of shape-invariant potentials is given by an infinite chain of reflectionless potentials $V_k^{(\pm)}(x)$ $(k = 0, 1, 2, \ldots)$, for which the associated superpotentials $W_k(x)$ satisfy the self-similar *ansatz* $W_k(x) = q^k W(q^k x)$, with $0 < q < 1$. These sets of partner potentials $V_k^{(\pm)}(x)$, also called *self-similar potentials* [29, 30], have their parameters related by a scaling: $a_n = a_1 q^{n-1}$. In the simplest case studied [31, 32], the remainder in the shape invariance condition (2) is given by $R(a_1) = ca_1$, where c is a constant. In these conditions, we obtain the remainder recurrence relation $R_n = R_1 q^{n-1}$, and using this relation in (5) and (20) and setting the energy scale as 1 making $R_1 = 1$, we find the eigenvalue factors e_n and $\mathcal{E}_r^{\{k\}}$ with the forms

$$e_n = 1 + q + q^2 + \cdots + q^{n-1} = \kappa_q(1-q^n) \qquad \text{and} \qquad \mathcal{E}_r^{\{k\}} = \kappa_q(q^{k-1} - q^r),$$
$$\text{where} \qquad \kappa_q = (1-q)^{-1}. \tag{69}$$

From expressions (69) for R_n, e_n and $\mathcal{E}_r^{\{k\}}$ and their relations with the shape-invariant systems generalized Stirling and Bell numbers through of equations (15), (16), (33), (35), (42), (58) and (59), we find these numbers given by polynomials with the form

$$\mathcal{S}_{r,s}^{(n)}(k, n-k) = q^{N(k)}\left[c_{r,s}^{(n,0)}(k) + c_{r,s}^{(n,1)}(k)q + c_{r,s}^{(n,2)}(k)q^2 + \cdots + c_{r,s}^{(n,m(k))}(k)q^{m(k)}\right] \tag{70}$$

$$\mathcal{B}_{r,s}(n) = \sum_{k=0}^{n-1} q^{N(k)}\left[c_{r,s}^{(n,0)}(k) + c_{r,s}^{(n,1)}(k)q + c_{r,s}^{(n,2)}(k)q^2 + \cdots + c_{r,s}^{(n,m(k))}(k)q^{m(k)}\right] \tag{71}$$

in such a way that their numerical values are dependent on the value of the scaling parameter q. For the $s = 1$ case, the power factors $N(k)$ and $m(k)$ have the values given by $N(k) = \frac{1}{2}\{(r-1)k(k+1)\}$ and $m(k) = (k-1)(n-k)$. Some examples illustrating this behaviour are

$$\mathcal{S}_{1,1}^{(5)}(5,0) = 1, \qquad \mathcal{S}_{1,1}^{(5)}(4,1) = 4 + 3q + 2q^2 + q^3, \qquad \mathcal{S}_{1,1}^{(5)}(3,2) = 6 + 8q + 7q^2 + 3q^3 + q^4,$$
$$\mathcal{S}_{1,1}^{(5)}(2,3) = 4 + 6q + 4q^2 + q^3, \qquad \mathcal{S}_{1,1}^{(5)}(1,4) = 1.$$

Table 4. Triangle of the generalized polynomial Stirling numbers $\mathcal{S}_{1,1}^{(n)}(k, n-k)$ for a self-similar potential system, given in the vector notation (72).

$r = 1,\ s = 1$	$\mathcal{S}_{1,1}^{(n)}(k, n-k), \quad 0 \le k \le n-1$					
$n = 1$	$[1]$					
$n = 2$	$[1]$	$[1]$				
$n = 3$	$[1]$	$\begin{bmatrix}2\\1\end{bmatrix}$	$[1]$			
$n = 4$	$[1]$	$\begin{bmatrix}3\\2\\1\end{bmatrix}$	$\begin{bmatrix}3\\3\\1\end{bmatrix}$	$[1]$		
$n = 5$	$[1]$	$\begin{bmatrix}4\\3\\2\\1\end{bmatrix}$	$\begin{bmatrix}6\\8\\7\\3\\1\end{bmatrix}$	$\begin{bmatrix}4\\6\\4\\1\end{bmatrix}$	$[1]$	
$n = 6$	$[1]$	$\begin{bmatrix}5\\4\\3\\2\\1\end{bmatrix}$	$\begin{bmatrix}10\\15\\16\\13\\7\\3\\1\end{bmatrix}$	$\begin{bmatrix}10\\20\\25\\19\\11\\4\\1\end{bmatrix}$	$\begin{bmatrix}5\\10\\10\\5\\1\end{bmatrix}$	$[1]$

Therefore, we conclude that the scaling parameter q defines families of Stirling and Bell numbers for the self-similar potentials. Since q defines the form of the self-similar potential, we then conclude that the form of the potential is responsible for the definition of the families of $\mathcal{S}_{r,s}^{(n)}(k, n-k)$ and $\mathcal{B}_{r,s}(n)$ numbers. We note that since q can assume any continuous value within of its physically allowed range, there is an infinite number of possible families of Stirling and Bell numbers for self-similar potential systems. The common x dependence of the potentials with different forms is responsible for the common set of polynomial coefficients $\{c_{r,s}^{(n,j)}(k);\ j = 0, 1, 2, \ldots, m(k)\}$ defining the elements $\mathcal{S}_{r,s}^{(n)}(k, n-k)$ and $\mathcal{B}_{r,s}(n)$ belong to different families (q values). Hence it is sufficient to specify a given Stirling number family to present the numerical values of the polynomial coefficients associated with these numbers. Therefore, after this point, we use the following vector notation:

$$\mathcal{S}_{r,s}^{(n)}(k, n-k) \Longleftrightarrow q^{N(k)} \begin{bmatrix} c_{r,s}^{(n,0)} \\ c_{r,s}^{(n,1)} \\ c_{r,s}^{(n,2)} \\ \vdots \\ c_{r,s}^{(n,m(k))} \end{bmatrix}. \tag{72}$$

From the power factor $N(k)$ and the number $m(k)$ of elements of the vector, we obtain the order of the polynomial for $\mathcal{S}_{r,s}^{(n)}(k, n-k)$. Tables 4, 5 and 6 present, in the vector notation

Table 5. The same as table 4 for the generalized polynomial Stirling numbers $\mathcal{S}^{(n)}_{2,1}(k, n-k)$.

$r = 2,\ s = 1$	$\mathcal{S}^{(n)}_{2,1}(k, n-k), \qquad 0 \le k \le n-1$				
$n = 1$	$[1]$				
$n = 2$	$[1]$	$q\begin{bmatrix}1\\1\end{bmatrix}$			
$n = 3$	$[1]$	$q\begin{bmatrix}1\\2\\2\\1\end{bmatrix}$	$q^3\begin{bmatrix}1\\2\\2\\1\end{bmatrix}$		
$n = 4$	$[1]$	$q\begin{bmatrix}1\\2\\3\\3\\2\\1\end{bmatrix}$	$q^3\begin{bmatrix}1\\3\\6\\8\\8\\6\\3\\1\end{bmatrix}$	$q^6\begin{bmatrix}1\\3\\5\\6\\5\\3\\1\end{bmatrix}$	
$n = 5$	$[1]$	$q\begin{bmatrix}1\\2\\3\\4\\4\\3\\2\\1\end{bmatrix}$	$q^3\begin{bmatrix}1\\3\\7\\12\\17\\20\\20\\17\\12\\7\\3\\1\end{bmatrix}$	$q^6\begin{bmatrix}1\\4\\10\\19\\29\\37\\40\\37\\29\\19\\10\\4\\1\end{bmatrix}$	$q^{10}\begin{bmatrix}1\\4\\9\\15\\20\\22\\20\\15\\9\\4\\1\end{bmatrix}$

(72), the polynomial giving the Stirling $\mathcal{S}^{(n)}_{r,s}(k, n-k)$ numbers for some values of the powers n, r and s belonging to a generic q-family of a self-similar potential system.

6. Final remarks

We solved the normal ordering problem for shape-invariant-system ladder operators $\hat{B}_+$ and $\hat{B}_-$ for strings of the form $(\hat{B}_+^r \hat{B}_-^s)^n$ and found that the solution involves expansion coefficient sequence which, for $r, s > 1$, corresponds to the generalization, for any shape-invariant potential system, of the classical Stirling and Bell numbers presented in the literature. We showed that the families of polynomials generated depend on the parameters related to the forms of the supersymmetric partner potentials. Since these parameters can assume any continuous value within the physical allowed range for them, there is an infinite number of possible families of Stirling and Bell numbers for these potential systems. We verified that the common x dependence of the potentials with different forms is responsible for the common set of polynomial coefficients that give the common structure for some families. The application of the general formalism to some shape-invariant potential systems shows that, in the case of

Table 6. The same as table 4 for the generalized polynomial Stirling numbers $\mathcal{S}^{(n)}_{3,2}(k, n-k)$.

$r=3,\ s=2$	$\mathcal{S}^{(n)}_{3,2}(k,n-k), \qquad 0 \le k \le 2(n-1)$				
$n=1$	$[1]$				
$n=2$	$[1]$	$q\begin{bmatrix}1\\2\\2\\1\end{bmatrix}$	$q^3\begin{bmatrix}1\\2\\2\\1\end{bmatrix}$		
$n=3$	$[1]$	$q\begin{bmatrix}1\\3\\4\\4\\3\\2\\1\end{bmatrix}$	$q^3\begin{bmatrix}2\\6\\12\\17\\19\\17\\12\\7\\3\\1\end{bmatrix}$	$q^3\begin{bmatrix}1\\4\\11\\26\\49\\75\\96\\104\\97\\79\\57\\37\\21\\10\\4\\1\end{bmatrix}$	$q^2\begin{bmatrix}1\\6\\23\\64\\140\\255\\402\\561\\700\\784\\793\\731\\621\\494\\374\\273\\191\\125\\74\\38\\16\\5\\1\end{bmatrix}$

the potentials presenting their parameters a_n related by a translation, it is possible to group the potentials in sets with a common polynomial structure for the Stirling and Bell numbers.

The framework developed in this paper suggests new possibilities in the theoretical study of the operator normal ordering problems, making the generalization of the definition of the Stirling and Bell numbers and the extension of the relations possible involving these new generalized numbers, such as Dobiński-type relations, generating functions, Sheffer-type identities, possible combinatorial relations, relations to quantum coherent states, etc. Also, other applications of this formalism to shape-invariant systems could be interesting. Among them we list as examples the study of forced systems, weakly coupled systems, driven shape-invariant potential systems, the generalization of the Wick's theorem for ladder operators, the study of driven damped systems, the generalization of the master equation, etc.

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